

Uncertainty in Measurements of Distance

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Abstract

Ng and van Dam have argued that quantum theory and general relativity give a lower bound $\Delta\ell \gtrsim \ell^{1/3}\ell_P^{2/3}$ on the uncertainty of any distance, where ℓ is the distance to be measured and ℓ_P is the Planck length. Their idea is roughly that to minimize the position uncertainty of a freely falling measuring device one must increase its mass, but if its mass becomes too large it will collapse to form a black hole. Here we show that one can go below the Ng–van Dam bound by attaching the measuring device to a massive elastic rod. Relativistic limitations on the rod’s rigidity, together with the constraint that its length exceeds its Schwarzschild radius, imply that zero-point fluctuations of the rod give an uncertainty $\Delta\ell \gtrsim \ell_P$.

1 Introduction

It has long been believed that quantum gravity effects become important at distances comparable to the Planck length, ℓ_P , and a number of arguments have been presented to support this idea [3, 4]. To show this sort of thing, one mainly needs to show that gravity effects become important at *some* fixed length scale depending only on the constants c , G and \hbar . Dimensional analysis does the rest, since ℓ_P is the only quantity with dimensions of length that one can construct from these constants. However, in situations where a second length scale becomes relevant, one cannot use dimensional analysis to settle all controversies. For example, Ng and van Dam [5] have recently argued that quantum gravity effects cause surprisingly large uncertainties in the measurement of a large distance ℓ , namely

$$\Delta\ell \gtrsim \ell^{1/3}\ell_P^{2/3}$$

where the symbol \gtrsim means that we are ignoring a constant factor of order unity. Amelino-Camelia [1] has gone even further, arguing that

$$\Delta\ell \gtrsim \ell^{1/2}\ell_P^{1/2}.$$

Uncertainties on this scale are on the brink of being experimentally detectable, lending extra interest to the issue. However, in what follows, we reanalyze the Ng–van Dam thought experiment and show that by modifying its design we can dramatically reduce the uncertainty of distance measurements. Our modified thought experiment gives

$$\Delta\ell \gtrsim \ell_P.$$

2 Ng–van Dam Thought Experiment

The elements of the Ng–van Dam thought experiment are straightforward, and the aim is to show that through a simple application of the uncertainty principle, together with limits imposed by general relativity, we are led to the conclusion that a fundamental distance uncertainty arises that may be far larger than the Planck scale.

The argument proceeds as follows. First consider two nearby objects in free fall approximately at rest relative to one another: an observer consisting of a clock and light emitter, and a mirror. If the observer wants to know the distance to the mirror, he may simply emit a burst of light, wait a time t for the light to return, and conclude that the mirror is a distance $\ell = ct/2$ away.

Now we are interested in the uncertainty of this measurement. Following an argument due to Wigner [7, 9] we treat the clock as a free quantum mechanical particle and impose the uncertainty condition $\Delta q \Delta p \gtrsim \hbar$. Writing $\Delta p = m \Delta v$ where m is the mass of the clock, we thus obtain the following bound on the uncertainty of the clock’s position at time t :

$$\begin{aligned} \Delta q(t) &= \Delta(q + tv) \\ &= \sqrt{(\Delta q)^2 + (t \Delta v)^2} \\ &\gtrsim \Delta q + \frac{\hbar t}{m \Delta q}. \end{aligned} \tag{1}$$

To minimize the position uncertainty at time t , we find that the optimal position uncertainty at time zero should be $\Delta q = \sqrt{\hbar t/m}$. Plugging this back into equation (1), we find the minimum uncertainty at time t to be

$$\Delta q(t) \gtrsim \sqrt{\hbar t/m}. \tag{2}$$

It is also convenient to write this in terms of the distance to be measured and the Compton wavelength of the clock, $\ell_C = \hbar/mc$:

$$\Delta q(t) \gtrsim \ell^{1/2} \ell_C^{1/2}. \tag{3}$$

This uncertainty in the position of the clock contributes to the uncertainty in ℓ , the distance between the clock and mirror. We can ignore the uncertainty in the position of the mirror, which behaves similarly, and obtain this lower bound on $\Delta \ell$:

$$\Delta \ell \gtrsim \ell^{1/2} \ell_C^{1/2}. \tag{4}$$

So far we have only considered the effects of quantum mechanics and the speed of light, with no mention of the effects of general relativity. Next, Ng and van Dam consider the details of the clock itself. They take the clock to consist of two parallel mirrors a distance d apart, and consider a tick of the clock to be the time $2d/c$ that it takes light to travel back and forth between them. Since we now have the length scale d and the mass scale m of the clock, we can now begin to consider general relativity effects. In particular, Ng and van Dam assert that the size of the clock, d , must be larger than its Schwarzschild radius $\ell_S = Gm/c^2$. If the tick of the clock is a lower bound on the accuracy of its time measurements, this requirement implies that

$$\Delta \ell \gtrsim \ell_S. \tag{5}$$

Finally, squaring the uncertainty from equation (4) and multiplying the result by equation (5), we obtain $(\Delta \ell)^3 \gtrsim \ell \ell_C \ell_S$. Note that $\ell_C \ell_S$ is equal to ℓ_P^2 , the Planck length squared. Thus the primary result obtained from the Ng–van Dam thought experiment is that the minimum uncertainty in this kind of measurement satisfies

$$\Delta \ell \gtrsim \ell^{1/3} \ell_P^{2/3}, \tag{6}$$

a bound depending only on the distance ℓ to be measured and the Planck length.

Ng and van Dam have suggested that this uncertainty is inherent in any distance measurement, and that it may be apparent in the latest generation of interferometers designed for detecting gravitational waves. As explained in Section 3, we believe that this conclusion is unwarranted, and that changes in the details of the clock used in the thought experiment will have an effect on the final uncertainty result, and thus lead to a breakdown of the Ng–van Dam uncertainty as a fundamental property of distance measurement.

Before turning to this we should comment a bit on the Wigner clock thought experiment. Our derivation of equation (2) is rigorous *if* in the second step we assume that the fluctuations in q and p are uncorrelated, or more precisely, that the expectation value $\langle pq + qp \rangle$ equals $2\langle p \rangle \langle q \rangle$. This is true if, for example, the wavefunction for the clock’s center of mass is initially Gaussian. If we allowed the wavefunction to be arbitrary, we could arrange for $\Delta q(t)$ to be arbitrarily small at some chosen time t , simply by taking a wavefunction close to a delta function at time t and evolving it backwards to get the wavefunction at time zero. However, for our purposes this counts as ‘cheating’, since we can only reduce $\Delta q(t)$ arbitrarily this way if we already know the time t that is to be measured.

3 A Modified Thought Experiment

As was pointed out by Adler, Nemenman, Ovenduin, and Santiago [2], the result of the above thought experiment is only valid when the clock evolves according to the free Schrödinger equation, and this need not be the case. Ng and van Dam [6] have, in turn, asserted that if the clock is bound through some potential, then it must be bound to something, and that this something can then be considered as a part of the clock. We will explore this line of reasoning through a specific example, and argue that attaching the clock to an external object does indeed affect the final uncertainty result.

For simplicity, let us take the clock to be attached to one end of a rod in free fall. We assume this rod has equilibrium length x , elastic constant Y , and mass m . We take the mass of the clock to be negligible compared to m . It is thus the rod’s mass, rather than that of the clock, which limits the spreading of the clock’s wavefunction with the passage of time. By making m large we can make this spread as slow as we like. However, special relativity puts a bound on the elastic constant Y , since a very rigid medium would allow sound waves to propagate with speed greater than that of light. This means that the ends of the rod will undergo zero-point fluctuations which also contribute to the uncertainty of any position we measure using the clock.

To calculate these, recall that the speed of sound along the rod will be $\sqrt{Yx/m}$. Since this must be less than c , we have

$$Y \leq mc^2/x. \quad (7)$$

Now, a one-dimensional rod with one end attached to an immovable wall can perform oscillations about its equilibrium length x exactly as a harmonic oscillator with spring constant $k = Y/x$ and oscillator mass $m_{osc} = m/3$. Thus, a rod in space will have a mode of oscillation that allows each end to perform harmonic oscillations about its respective equilibrium position with equivalent spring constant $k = 2Y/x$ and oscillator mass $m_{osc} = m/6$. Treating the problem quantum-mechanically, the ground state of such an oscillator has position uncertainty

$$\Delta x = \sqrt{\frac{\hbar}{2(km_{osc})^{1/2}}} = \sqrt{\frac{\hbar}{2} \left(\frac{3x}{mY} \right)^{1/2}} \quad (8)$$

Using equation (7) and dropping factors of order unity, we obtain

$$\Delta x \simeq \sqrt{\hbar x/mc} \quad (9)$$

There are other states where Δx is smaller at one moment, but not for the entire duration of the experiment, at least if the rod’s period of oscillation is short compared to the time t which the clock

is to measure — and if it were not, the rod would fail to slow the spread of the clock’s wavefunction. We may thus take this value of Δx as an approximate lower bound on the uncertainty of the clock’s position with respect to rod’s center of mass. In terms of the Compton wavelength of the rod, it follows that

$$\Delta x \gtrsim x^{1/2} \ell_C^{1/2}. \quad (10)$$

The rod’s center of mass will also have an uncertainty in its position, given by equation (3). Both this and Δx will contribute to the uncertainty of any distance we measure by sending a burst of light from the clock and measuring the time it takes for the light to return. Assuming these uncertainties are uncorrelated, we obtain

$$\Delta \ell \gtrsim (\ell^{1/2} + x^{1/2}) \ell_C^{1/2}. \quad (11)$$

Next we must rethink the requirements imposed by general relativity on the clock-rod system. The requirement made in the original Ng-van Dam thought experiment was that the clock must be larger than its own Schwarzschild radius. While this is still true, it is not important now, since we no longer need the clock to be heavy to prevent its wave function from spreading with time. Instead, the important relativity consideration is that the rod be longer than its Schwarzschild radius:

$$x \gtrsim \ell_S \quad (12)$$

Although we do not have spherical symmetry in this example, the Schwarzschild radius remains useful as a quick and dirty estimate of the length scale at which the rod would collapse to form a black hole. Indeed, the ‘hoop conjecture’ [8] says roughly that *any* system compressed within its Schwarzschild radius must give rise to a singularity.

Plugging this lower bound on x into equation (10), we obtain

$$\Delta \ell \gtrsim (\ell^{1/2} + \ell_S^{1/2}) \ell_C^{1/2}. \quad (13)$$

We can minimize this uncertainty by making the rod very heavy, so that $\ell_C \rightarrow 0$, leaving us with

$$\Delta \ell \gtrsim \ell_P. \quad (14)$$

It is interesting to compare this argument to that given by Ng and van Dam. First, unlike their argument, ours does not assume any limitation on the clock’s accuracy solely due to its size. Second, while their result was obtained by multiplying two independent lower bounds on $\Delta \ell$, one from quantum mechanics and the other from general relativity, ours arises from an interplay between competing effects. On the one hand, we wish to make the rod as heavy as possible to minimize the quantum-mechanical spreading of its center of mass. To prevent it from becoming a black hole, we must also make it very long. On the other hand, as it becomes longer, the zero-point fluctuations of its ends increase, due to the relativistic limitations on its rigidity. We achieve the best result by making the rod just a bit longer than its own Schwarzschild radius.

We should also mention and rebut a possible objection to our argument. One could try to eliminate the effect of zero-point fluctuations on the position of the clock by attaching the clock to the midpoint of the rod rather than the end. This would indeed eliminate the effect of the vibrational mode we have been considering. However, there are other modes in which the midpoint of the rod oscillates. If we take these into account, some of the numbers in our formulas change, but we again find that $\Delta \ell$ is greater than ℓ_P times some constant of order unity.

4 Conclusions

It appears that by attaching the clock used in the Ng-van Dam thought experiment to a heavy object, the surprisingly large uncertainties arising in distance measurements can be avoided. It is

thus our opinion that these large uncertainties cannot be a fundamental feature of the union of quantum mechanics and general relativity.

Following Ng and van Dam, most recent papers on this topic suggest laser interferometry as a possible way to actually detect the uncertainties in measurements of distance caused by quantum gravity effects [1, 2, 3, 5, 6]. While this is tempting, it is worth noting that interferometers do not measure distances so much as periodic changes in distance. Indeed, the design of an interferometer is very different from the Ng–van Dam thought experiment or our modified version, so any attempt to detect quantum gravity effects at LIGO will require a separate analysis.

References

- [1] G. Amelino-Camelia, Quantum theory’s last challenge, *Nature* **408** (2000), 661–664.
- [2] R. J. Adler, I. M. Nemenman, J. M. Overduin and D. I. Santiago, On the detectability of quantum spacetime foam with gravitational-wave detectors, *Phys. Lett.* **B477** (2000), 424–428.
- [3] R. J. Adler and D. I. Santiago, On gravity and the uncertainty principle, *Mod. Phys. Lett.* **A14** (1999), 1371–1381.
- [4] C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation*, Freeman, New York, 1973, §43.4.
- [5] Y. J. Ng and H. van Dam, Measuring the foaminess of space-time with gravity-wave interferometers, *Found. Phys.* **30** (2000), 795–805.
- [6] Y. J. Ng and H. van Dam, On Wigner’s clock and the detectability of spacetime foam with gravitational-wave interferometers, *Phys. Lett.* **B477** (2000), 429–435.
- [7] H. Salecker and E. P. Wigner, Quantum limitations of the measurement of space-time distances, *Phys. Rev.* **109** (1958), 571–577.
- [8] K. S. Thorne, Nonspherical gravitational collapse: a short review, in *Magic without Magic*, ed. J. R. Klauder, Freeman, San Francisco, 1972, pp. 231–258.
- [9] E. P. Wigner, Relativistic invariance and quantum phenomena, *Rev. Mod. Phys.* **29** (1957), 255–268.